

6 SUSY QCD Part II

6.1 Flat Directions (Classical Moduli Space)

Recall

$$D^a = g(\phi^{*in}(T^a)_n^m \phi_{mi} - \bar{\phi}^{in}(T^a)_n^m \bar{\phi}_{mi}^*) \quad (6.1)$$

and the potential is:

$$V = \frac{1}{2} D^a D^a . \quad (6.2)$$

Define

$$D_m^n \equiv \langle \phi^{*in} \phi_{mi} \rangle \quad (6.3)$$

$$\bar{D}_m^n = \langle \bar{\phi}^{in} \bar{\phi}_{mi}^* \rangle \quad (6.4)$$

D_m^n and \bar{D}_m^n are $N \times N$ positive semi-definite Hermitian matrices of rank F . In a vacuum state we must have:

$$D^a = T_n^{am} (D_m^n - \bar{D}_m^n) = 0 \quad (6.5)$$

Since T^a is a complete basis for traceless matrices, we must have

$$D_m^n - \bar{D}_m^n = \alpha I \quad (6.6)$$

D_m^n can be diagonalized by an $SU(N)$ gauge transformation

$$U^\dagger D U \quad (6.7)$$

There will be at least $N - F$ zero eigenvalues, while the rest are positive semi-definite.

$$D = \begin{pmatrix} v_1^2 & & & & & \\ & v_2^2 & & & & \\ & & \ddots & & & \\ & & & v_F^2 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \quad (6.8)$$

where $v_i^2 \geq 0$. In this basis \overline{D}_m^n must also be diagonal, and it must also have $N - F$ zero eigenvalues. This tells us that $\alpha = 0$, and hence that

$$\overline{D}_m^n = D_m^n \quad (6.9)$$

D_m^n and \overline{D}_m^n are invariant under flavor transformations since

$$\phi_{mi} \rightarrow \phi_{mi} V_j^i \quad (6.10)$$

$$D_m^n \rightarrow V_i^{*j} \langle \phi^{*in} \rangle \langle \phi_{mi} \rangle V_j^i \quad (6.11)$$

$$\rightarrow \langle \phi^{*jn} \phi_{mj} \rangle = D_m^n \quad (6.12)$$

Thus, up to a flavor transformation we can write

$$\langle \overline{\phi}^* \rangle = \langle \phi \rangle = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad (6.13)$$

So we see that the D-term potential has flat directions emanating from the zero energy vacuum at $\phi = 0$, $\overline{\phi} = 0$, or in other words there is a space of degenerate vacua. This space is referred to as a *moduli space* since there are some massless fields (moduli fields) associated with it. As we change the values of the vevs we move between physically different vacua with different particle spectra.

At a generic point in the moduli space the $SU(N)$ gauge symmetry is broken to $SU(N - F)$.

6.2 SuperHiggs Mechanism?

Consider the simple case when $v_1 = v$ and $v_i = 0$, for $i > 1$. Then the gauge symmetry breaks from $SU(N)$ to $SU(N - 1)$ and the non-Abelian flavor symmetry breaks from $SU(F) \times SU(F)$ to $SU(F - 1) \times SU(F - 1)$. The number of broken gauge generators is $N^2 - 1 - ((N - 1)^2 - 1) = 2(N - 1) + 1$. A convenient basis of gauge generators for describing this broken gauge theory is given by $G^A = X^0, X_1^m, X_2^m, T^a$ where $A = 1, \dots, N^2 - 1$, $m = 1, \dots, N - 1$, and $a = 1, \dots, (N - 1)^2 - 1$. The X 's are the broken generators

while the T 's are the unbroken $SU(N-1)$ generators. The X 's are analogues of the Pauli matrices:

$$X^0 = \frac{1}{\sqrt{2(N^2 - N)}} \begin{pmatrix} N-1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} \quad (6.14)$$

$$X_1^m = \frac{1}{2} \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \\ 1 & & & \mathbf{0} & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix} \quad (6.15)$$

$$X_2^m = \frac{1}{2} \begin{pmatrix} 0 & \dots & 0 & i & 0 & \dots & 0 \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \\ -i & & & \mathbf{0} & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix} \quad (6.16)$$

Where only the $(1, m+1)$ and $(m+1, 1)$ components of X_1^m and X_2^m are non-zero. We can also make raising and lowering operators:

$$X^{+m} = \frac{1}{\sqrt{2}}(X_1^m - iX_2^m) \quad (6.17)$$

$$X^{-m} = \frac{1}{\sqrt{2}}(X_1^m + iX_2^m) \quad (6.18)$$

$$X^{+m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ & & & \mathbf{0} & & & \end{pmatrix} \quad (6.19)$$

$$X^{-m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -i & \mathbf{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (6.20)$$

We can then write the product of two generators (without a contraction of row and column indices) as:

$$G^A G^A = X^0 X^0 + X^{+m} X^{-m} + X^{-m} X^{+m} + T^a T^a \quad (6.21)$$

Rewriting

$$\phi \rightarrow \langle \phi \rangle + \phi \quad (6.22)$$

we have

$$\sum_A G^A \langle \phi \rangle = X^0 \langle \phi \rangle + X^{-m} \langle \phi \rangle \quad (6.23)$$

$$\langle \phi \rangle \sum_A G^A = \langle \phi \rangle X^0 + \langle \phi \rangle X^{+m} \quad (6.24)$$

We can label the components of the gluino field as

$$G^A \lambda^A = X^0 \Lambda^0 + X^{+m} \Lambda^{+m} + X^{-m} \Lambda^{-m} + T^a \lambda^a \quad (6.25)$$

and the quark field as

$$Q = \begin{pmatrix} \omega^0 & Q''_i \\ \omega^m & Q'_{mi} \end{pmatrix} \quad (6.26)$$

$$\bar{Q} = \begin{pmatrix} \bar{\omega}^0 & \bar{\omega}^m \\ \bar{Q}''_i & \bar{Q}'_{im} \end{pmatrix} \quad (6.27)$$

where Q' is a matrix with $N - 1$ rows and $F - 1$ columns.

We can then write the fermion mass terms generated by the Yukawa interactions as

$$\mathcal{L}_{\text{F mass}} = -\sqrt{2}g \left[\left(\langle \phi^* \rangle X^0 \Lambda^0 + \langle \phi^* \rangle X^{+m} \Lambda^{+m} \right) Q \right. \quad (6.28)$$

$$\left. -\bar{Q} \left(X^0 \Lambda^0 \langle \bar{\phi}^* \rangle + X^{-m} \Lambda^{-m} \langle \bar{\phi}^* \rangle \right) + h.c. \right] \quad (6.29)$$

$$= -gv \left[\sqrt{\frac{N-1}{N}} \left(\omega^0 \Lambda^0 - \bar{\omega}^0 \Lambda^0 \right) \right. \quad (6.30)$$

$$\left. + \omega^m \Lambda^{+m} - \bar{\omega}^m \Lambda^{-m} + h.c. \right] \quad (6.31)$$

So we have a Dirac fermion $(\Lambda^0, \frac{1}{\sqrt{2}}(\omega^0 - \bar{\omega}^0))$ with mass $gv\sqrt{\frac{2(N-1)}{N}}$, two Dirac fermions (Λ^{+m}, ω^m) , $(\Lambda^{-m}, -\bar{\omega}^m)$ with mass gv , and massless Weyl fermions Q' , \bar{Q}' , Q'' , \bar{Q}'' , and $\frac{1}{\sqrt{2}}(\omega^0 + \bar{\omega}^0)$.

$$\phi = \begin{pmatrix} h & \phi_i'' \\ H^m & \phi_{mi}' \end{pmatrix} \quad (6.32)$$

$$\bar{\phi} = \begin{pmatrix} \bar{h} & \bar{H}^m \\ \bar{\phi}_i'' & \bar{\phi}_{im}' \end{pmatrix} \quad (6.33)$$

where ϕ' is a matrix with $N-1$ rows and $F-1$ columns.

$$V_{\text{mass}} = \frac{g^2}{2} \left[\langle \phi^* \rangle (X^0 + X^{+m}) \phi + \phi^* (X^0 + X^{-m}) \langle \phi \rangle \right. \quad (6.34)$$

$$\left. - \langle \bar{\phi} \rangle (X^0 + X^{+m}) \bar{\phi}^* - \bar{\phi} (X^0 + X^{-m}) \langle \bar{\phi}^* \rangle \right] \quad (6.35)$$

$$\frac{g^2}{2} \left[\frac{(N-1)^2}{2(N^2-N)} \left(h + h^* - (\bar{h}^* + \bar{h}) \right)^2 \right. \quad (6.36)$$

$$\left. + (H^m - \bar{H}^{*m})(H^{*m} - \bar{H}^m) \right] \quad (6.37)$$

Choose a new basis for the scalar field that diagonalizes the mass matrix:

$$H^{+m} = \frac{1}{\sqrt{2}}(H^m - \bar{H}^{*m}) \quad (6.38)$$

$$H^{-m} = \frac{1}{\sqrt{2}}(H^{*m} - \bar{H}^m) \quad (6.39)$$

$$\pi^{+m} = \frac{1}{\sqrt{2}}(H^m + \bar{H}^{*m}) \quad (6.40)$$

$$\pi^{-m} = \frac{1}{\sqrt{2}}(H^{*m} + \bar{H}^m) \quad (6.41)$$

$$h^0 = \text{Re}(h - \bar{h}) \quad (6.42)$$

$$\pi^0 = \text{Im}(h - \bar{h}) \quad (6.43)$$

$$\Omega = \frac{1}{\sqrt{2}}(h + \bar{h}) \quad (6.44)$$

$$(6.45)$$

So

$$V_{\text{mass}} = g^2 v^2 \left[\frac{N-1}{N} (h^0)^2 + H^{+m} H^{-m} \right] \quad (6.46)$$

Thus we have a real scalar h^0 with mass $gv\sqrt{2(N-1)/N}$ a complex scalar H^{+m} (and its conjugate H^{-m}) with mass gv and a massless complex scalar Ω . The π 's will form the longitudinal components of the massive gauge bosons. They can be removed by going to Unitary gauge (i.e. by performing a gauge transformation $\exp(iX \cdot \pi)$).

We can write the gauge fields as:

$$G^B A_\mu^B = X^0 W_\mu^0 + X^{+m} W_\mu^{+m} + X^{-m} W_\mu^{-m} + T^a A_\mu^a \quad (6.47)$$

$$\begin{aligned} \mathcal{L}_{A^2\phi^2} &= g^2 A_\mu^A A_\nu^B g^{\mu\nu} \langle \phi^* \rangle G^A G^B \langle \phi \rangle \\ &= g^2 g^{\mu\nu} \langle \phi^* \rangle (X^0 W_\mu^0 X^0 W_\nu^0 + X^{+m} W_\mu^{+m} X^{-m} W_\nu^{-m} + X^{-m} W_\mu^{-m} X^{+m} W_\nu^{+m}) \langle \phi \rangle \\ &= g^2 v^2 g^{\mu\nu} \left(\frac{N-1}{2N} W_\mu^0 W_\nu^0 + \frac{1}{2} W_\mu^{+m} W_\nu^{-m} \right) \end{aligned} \quad (6.48)$$

Since there is an identical term arising from $\mathcal{L}_{A^2\bar{\phi}^2}$ we have a gauge boson W_μ^0 with mass $gv\sqrt{2(N-1)/N}$, a pair of gauge bosons W_μ^{+m} and W_μ^{-m} with mass gv , and the massless gauge bosons A_μ^a of $SU(N-1)$. As expected all the particles fall into supermultiplets.

To summarize: for $v = 0$ we have the massless fields:

	$SU(N)$	$SU(F)$	$SU(F)$
Q	\square	\square	$\mathbf{1}$
\bar{Q}	$\bar{\square}$	$\mathbf{1}$	$\bar{\square}$

for $v \neq 0$ we have massive states:

	$SU(N-1)$	$SU(F-1)$	$SU(F-1)$
W^0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
W	\square	$\mathbf{1}$	$\mathbf{1}$
\overline{W}	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$

Where the massive vector supermultiplet $W^0 = (W_\mu^0, h^0, \Lambda^0, \frac{1}{\sqrt{2}}(\omega^0 - \bar{\omega}^0))$ has mass

$$m_{W^0} = gv \sqrt{\frac{2(N-1)}{N}} \quad (6.49)$$

and the massive vector supermultiplets $W^{+m} = (W_\mu^{+m}, H^{+m}, \Lambda^{+m}, \omega^m)$ and $W^{-m} = (W_\mu^{-m}, H^{-m}, \Lambda^{-m}, \bar{\omega}^m)$ have mass

$$m_{W^\pm} = gv. \quad (6.50)$$

We also have the massless states:

	$SU(N-1)$	$SU(F-1)$	$SU(F-1)$
Q'	\square	\square	$\mathbf{1}$
\overline{Q}'	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$
Q''	$\mathbf{1}$	\square	$\mathbf{1}$
\overline{Q}''	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Where the singlet chiral supermultiplet S is $(\frac{1}{\sqrt{2}}(h + \bar{h}), \frac{1}{\sqrt{2}}(\omega^0 + \bar{\omega}^0))$. Including the gluons (and gluinos) we have for both cases ($v = 0$ and $v \neq 0$) $2(N^2 - 1) + 4FN$ boson degrees of freedom (and of course the same number of fermion degrees of freedom).

References

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